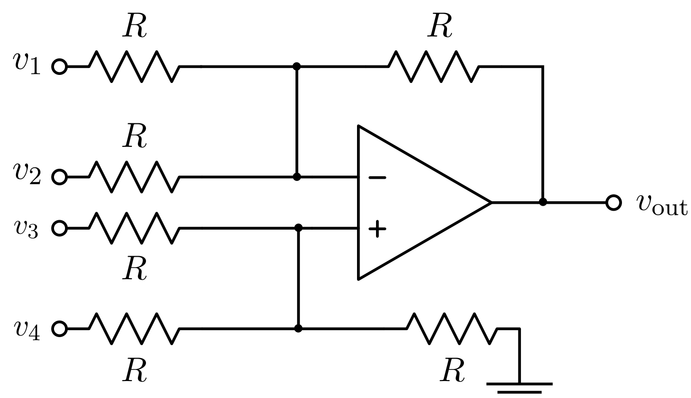
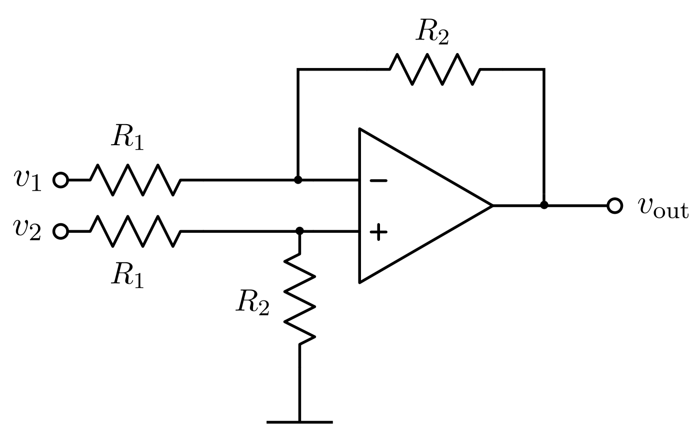
PHYS 231 – Assignment #4

Due Monday, Nov. 20 @ 10:00 am

1. Show that the output of the circuit below is $v_{\text{out}} = v_3 + v_4 - v_1 - v_2$. Derive expressions for the voltages v_- and v_+ at the two the op amp inputs. Show your work.

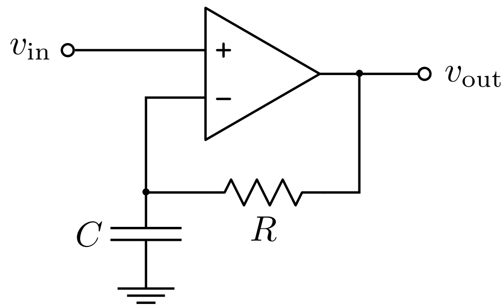


2. Show that the circuit below is a differential amplifier. That is, show that $v_{\text{out}} = G\,(v_2 - v_1)$. Find an expression for G in terms of R_1 and R_2. Simplify your answers as much as possible and show all of your work.



3. Consider the op-amp circuit below. (a) Show that the output voltage v_{out} can be expressed in terms of the following first-order differential equation:

$$v_{\text{out}} = v_{\text{in}} + RC\frac{dv_{\text{in}}}{dt} \tag{1}$$



(b) Now suppose that v_{in} is sinusoidal such that it can be expressed as $v_{\text{in}} = V_{\text{i}}e^{j\omega t}$. In this case the output will be of the form $v_{\text{out}} = V_0 e^{j(\omega t+\phi)}$. Substitute the exponential forms of v_{in} and v_{out} into Eq. 1 and find expressions for the amplitude V_0 and phase ϕ of the output voltage. Your expressions should be in terms of V_{i}, ω, R, and C.

(c) We know that for sinusoidal signals we can express the voltage across the capacitor as $v_{\text{C}} = iZ_{\text{c}}$ where $Z_{\text{C}} = 1/(j\omega C)$. We also know that the circuit above has the form of a non-inverting amplifier such that:

$$v_{\text{out}} = \left(1 + \frac{R}{Z_{\text{C}}}\right) v_{\text{in}} \tag{2}$$

If v_{in} has amplitude V_{i}, use Eq. 2 to find the amplitude and phase of v_{out}.

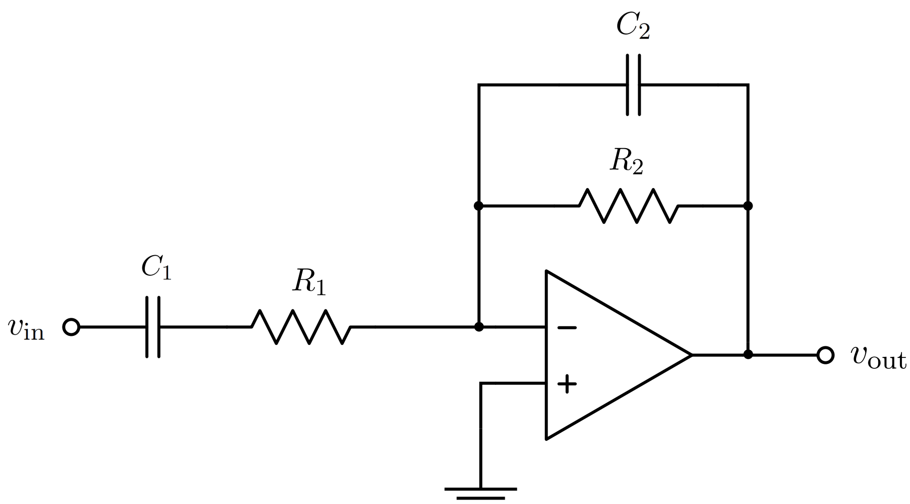
The following problems are for your own practice. *They won't be graded.*

Eggleston Chapter 6 #1, 3, 5, 8(a), 8(b), 8(d).

1. The circuit below is a *bandpass* filter/amplifier. Show that:

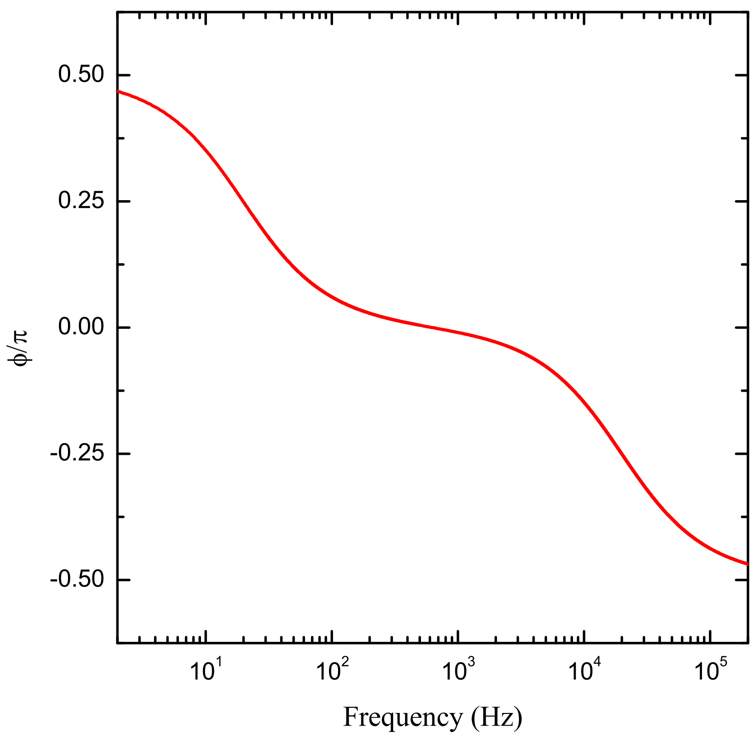
$$\left|\frac{v_{\text{out}}}{v_{\text{in}}}\right| = G\frac{\omega R_1 C_1}{\sqrt{1+(\omega R_1 C_1)^2}}\frac{1}{\sqrt{1+(\omega R_2 C_2)^2}}.$$

Assume that v_{in} is a sinusoidal input. When analysing the circuit, make use of the fact that $Z_C = 1/(j\omega C)$ and $v_C = iZ_C$. What is G in terms of the circuit components?



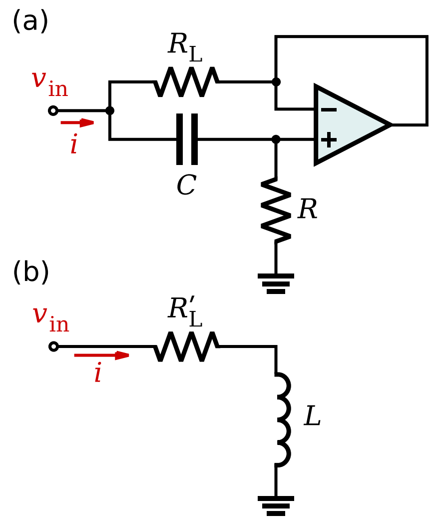
For $R_1 = 10$ kΩ, $C_1 = 0.8$ μF, $R_2 = 100$ kΩ, and $C_2 = 80$ pF, plot $|v_{\text{out}}/v_{\text{in}}|$ as a function of frequency from $f = 2$ Hz to 200 kHz on a log-log scale (recall that $\omega = 2\pi f$). You should find that the gain of the amplifier is relatively constant between the 20 Hz and 20 kHz, but drops quickly outside of this range. This circuit could be used as a basic audio amplifier.

For the audio filter/amplifier above, derive an expression for the phase shift ϕ of v_{out} relative to v_{in}. Plot ϕ as a function of frequency from 2 Hz to 200 kHz with the frequency axis on a log scale. If done correctly, you should have reproduced the plot below.



Notice that the phase shift is relatively small near the centre of the audio frequency range, but changes rapidly near 20 Hz and 20 kHz. Good audio amplifiers are designed to have both a flat gain and minimal phase shift over the entire the audio frequency range.

2. Circuit (a) in the figure below can be used to simulate an inductance. In this problem you will attempt to show that, under certain conditions, the impedance Z_a of circuit (a) is approximately equivalent to the impedance Z_b of circuit (b).



(i) First, calculate the effective impedance of the op-amp circuit using $Z_\mathrm{a} = v_\mathrm{in}/i$. Specifically, show that:

$$Z_\mathrm{a} = R_\mathrm{L}\frac{1 + j\omega RC}{1 + j\omega R_\mathrm{L}C}.$$

(ii) Show that Z_a can be re-expressed as:

$$Z_\mathrm{a} = R_\mathrm{L}\left[\frac{1 + \omega^2 RR_\mathrm{L}C^2}{1 + (\omega R_\mathrm{L}C)^2}\right] + j\omega\left[\frac{R_\mathrm{L}\,(R - R_\mathrm{L})\,C}{1 + (\omega R_\mathrm{L}C)^2}\right]$$

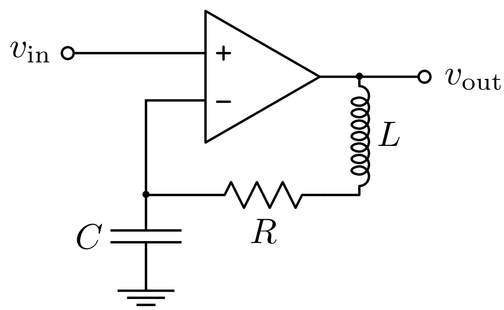
The series combination of R'_L and $Z_L = j\omega L$ shown in part (b) of the figure is an equivalent circuit with impedance given by $Z_\mathrm{b} = R'_\mathrm{L} + j\omega L$.

(iii) Consider a frequency $f = 100$ Hz, $R_\mathrm{L} = 10\ \Omega$, $R = 10\ \mathrm{k}\Omega$, and $C = 1\ \mu\mathrm{F}$. What are the effective resistance R'_L and inductance L of the op-amp circuit? Give numerical values in terms of ohms and henries respectively. Suppose you wanted to wind a coil of wire to make the equivalent inductance. If the inductor length is 2 cm and its diameter is 1 cm, how many turns of wire would be required?

The op-amp circuit shown above can be used to simulate a large inductance that would be impractical to make by winding a coil of wire. For the component values chosen in part (iii), $\omega R_\mathrm{L}C \ll 1$ and $\omega^2 RR_\mathrm{L}C^2 \ll 1$ such that $R'_\mathrm{L} \approx R_\mathrm{L}$. Furthermore, $R_\mathrm{L} \ll R$ such that $L \approx R_\mathrm{L}RC$.

3. Consider the op-amp circuit below. (a) Show that the output voltage v_{out} can be expressed in terms of the following second-order differential equation:

$$v_{\text{out}} = v_{\text{in}} + RC\frac{dv_{\text{in}}}{dt} + LC\frac{d^2 v_{\text{in}}}{dt^2} \tag{3}$$



(b) Now suppose that v_{in} is sinusoidal such that it can be expressed as $v_{\text{in}} = V_{\text{i}}e^{j\omega t}$. In this case the output will be of the form $v_{\text{out}} = V_0 e^{j(\omega t+\phi)}$. Substitute the exponential forms of v_{in} and v_{out} into Eq. 3 and find expressions for the amplitude V_0 and phase ϕ of the output voltage. Your expressions should be in terms of V_{i}, ω, R, and C.

(c) We know that for sinusoidal signals we can express the voltage across the capacitor as $v_{\text{C}} = iZ_{\text{c}}$ and the voltage across the inductor as $v_{\text{L}} = iZ_{\text{L}}$ where $Z_{\text{C}} = 1/(j\omega C)$ and $Z_{\text{L}} = j\omega L$. We also know that the circuit above has the form of a non-inverting amplifier such that:

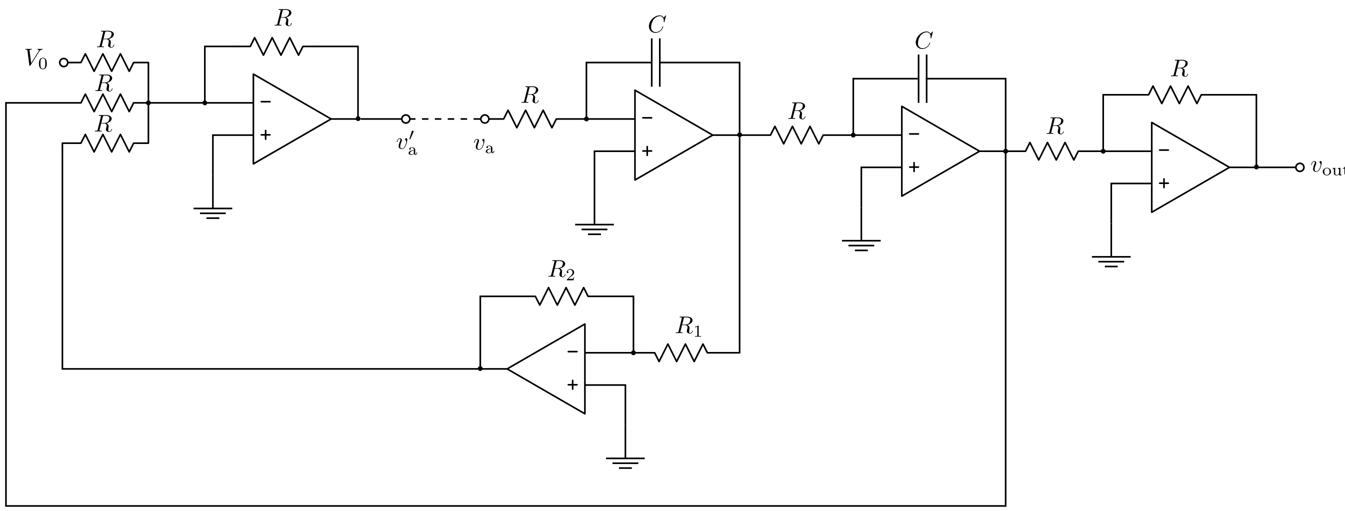
$$v_{\text{out}} = \left(1 + \frac{R + Z_{\text{L}}}{Z_{\text{C}}}\right) v_{\text{in}} \tag{4}$$

If v_{in} has amplitude V_{i}, use Eq. 4 to find the amplitude and phase of v_{out}.

4. In this problem we attempt to show that the output of the op-amp circuit below is the solution to a second-order differential equation. The circuit consists of five op-amp sub-circuits: one summing amplifier, two integrators, and two inverting amplifiers. To see that v_{out} must be the solution to a differential equation, start by assuming that the voltage v_{a} can, somehow, be set to:

$$v_{\text{a}} = -\left(RC\right)^2 \frac{d^2 v}{dt^2} \tag{5}$$

We will see how this can be done very shortly.



(a) Confirm that, if v_{a} is given by Eq. 5, then $v_{\text{out}} = v$ and:

$$v'_{\text{a}} = RC\frac{R_2}{R_1}\frac{dv}{dt} + v - V_0 \tag{6}$$

where V_0 is assumed to be a constant.

(b) Now assume that the circuit nodes labeled v'_{a} and v_{a} are connected by a wire (as indicated by the dashed line). Show that this leads to the requirement that v satisfy the following differential equation:

$$\frac{d^2 v}{dt^2} + \frac{1}{RC}\frac{R_2}{R_1}\frac{dv}{dt} + \frac{1}{\left(RC\right)^2}v = \frac{V_0}{\left(RC\right)^2} \tag{7}$$

Therefore, by summing the appropriate integrals of v_{a}, we can construct a differential equation of any order.

(c) Recall that we saw a similar differential equation when solving for the transient charge on a capacitor in a series LRC circuit:

$$\frac{d^2q}{dt^2} + \gamma\frac{dq}{dt} + \omega_0^2 q = \frac{V_0}{L} \tag{8}$$

where $\gamma = R/L$ and $\omega_0 = 1/\sqrt{LC}$. For the so-called underdamped case ($\gamma \ll \omega_0$), we found that the charge on the capacitor exhibited damped oscillations where the oscillation frequency was, to a good approximation, equal to $\omega_0/(2\pi)$ and the damping time constant was given by $\tau = 2/\gamma$.

Confirm that the requirement for underdamping in Eq. 6 is that $R_1 \gg R_2$. If this condition is satisfied, find expressions for the oscillation frequency and damping time constant for the voltage v.

If $R_1 = 100\,\text{k}\Omega$, $R_2 = R = 10\,\text{k}\Omega$, and $C = 1\,\text{nF}$ what are the numerical values for the oscillation frequency and damping time constant for v?